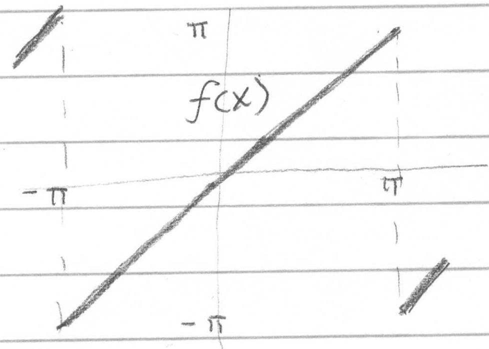
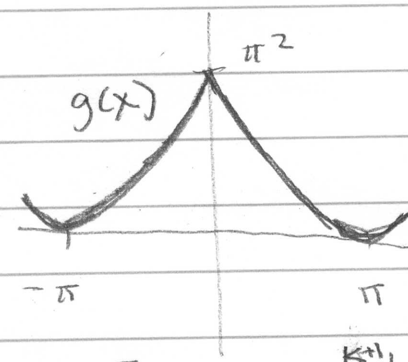
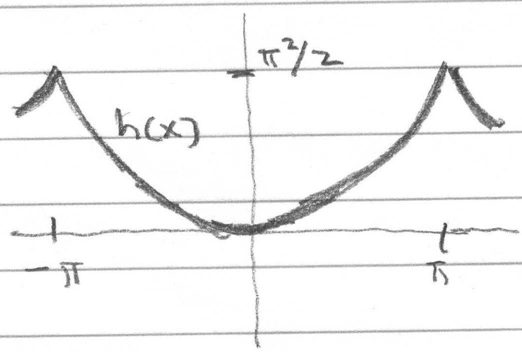




use the FS $f(x) = \sum_{k\geq 1} 2(-1)^{k+1}\frac{1}{k}\sin(kx)$
to get a FS for $g(x)$.

$$h(x) = \int_0^x f(y)dy$$
$$= \int_0^y \sum_{k\geq 1} 2(-1)^{k+1}\frac{1}{k}\sin(ky)dy$$
$$= \sum_{k\geq 1} 2(-1)^{k+1}\frac{1}{k}\left[\frac{-\cos(ky)}{k}\right]_{y=0}^{x}$$
$$= \sum_{k=1}^{\infty} \frac{2(-1)^{k+2}}{k^2}(\cos(kx)-1)$$
$$= \left(\sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k^2}\right) + \sum_{k=1}^{\infty} \frac{2(-1)^k}{k^2}\cos(kx)$$



Notice that
$g(x) = 2h(x+\pi)$
(I got the factor of 2 since $g(0)=\pi^2$ and $h(\pi) = \int_0^\pi x\,dx = \pi^2/2$)

$$g(x) = 2\left(\underbrace{\left(\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{k^2}\right)}_{=\pi^2/6} + \sum_{k=1}^{\infty}\frac{2(-1)^k}{k^2}\cos(k(x+\pi))\right)$$
$$= \pi^2/3 + \sum_{k=1}^{\infty}\frac{4}{k^2}\cos(kx)$$

(If you didn't know the sum value $\pi^2/6$ that's ok)

2 With respect to what modes do the following FS converge

2a $f(x) = \sin(1/x) \qquad 0 \le x \le 2\pi$

Since $-1 \le f(x) \le 1$ we see that
$$\|f\|_2 = \left(\int_0^{2\pi} |f(x)|^2\,dx\right)^{1/2} \le \left(\int_0^{2\pi} 1\,dx\right)^{1/2} < \infty$$

So $f(x) \in L^2[0, 2\pi]$. Therefore $S_n f \to f$ in L^2-norm

Since $f(x)$ has infinitely many oscillations near $x = 0$, it is not piecewise continuous. Dirichlet's theorems do not imply uniform or pointwise convergence for $f(x)$.

2b $g(x) = x^{-1/2} \qquad 0 \le x \le 2\pi$

$$\|g\|_2 = \left(\int_0^{2\pi} |x^{-1/2}|^2\,dx\right)^{1/2} = \left(\int_0^{2\pi} x^{-1}\,dx\right)^{1/2} = \infty$$

So $g \notin L^2[0, 2\pi]$. Then the L^2-convergence theorem doesn't apply to g.

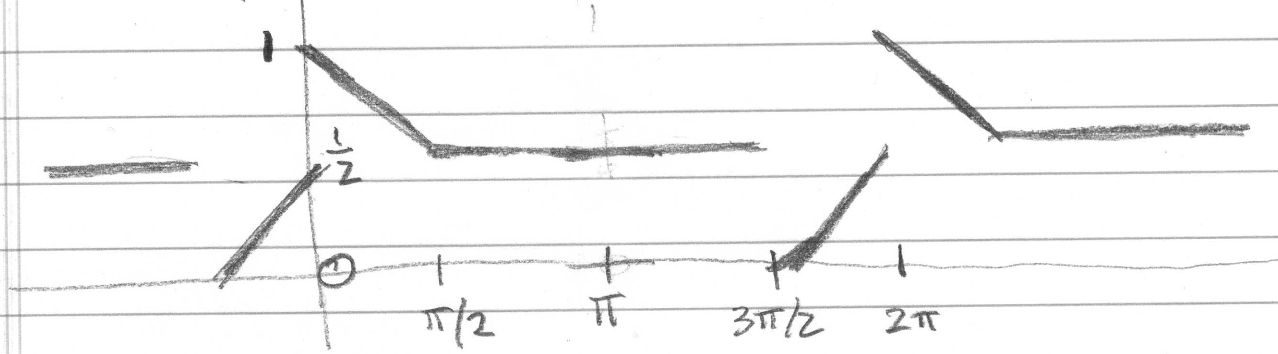
Since $g(x) \to \infty$ as $x \to 0$, g isn't piecewise continuous, so Dirichlet's theorems don't apply.

2c $h(x) = |x-\pi|, \quad 0 \le x \le 2\pi$

This function is bounded, so $h \in L^2[0,2\pi]$. Thus $S_n h \to h$ in L^2-norm

Since $h(x)$ is piecewise smooth and continuous, Dirichlet's theorems tell us that $S_n h \to h$ pointwise and uniformly.

3



This function f has $S_n f(x) \to f(x)$ for all x values except the jumps $x = 2k\pi$, $x = 3\pi/2 + 2k\pi$ $(k \in \mathbb{Z})$ (By Dirichlet's pointwise theorem)

At each jump it has an overshoot:

$$\text{Overshoot}(S_n f, 0) \approx 0.0895 \cdot |\text{Jump}(f,0)|$$
$$= 0.0895 \cdot 1/2$$
$$\approx 0.0448$$

$$\text{Overshoot}(S_n f, \tfrac{3\pi}{2}) \approx 0.0895 \cdot |\text{Jump}(f, \tfrac{3\pi}{2})|$$
$$= 0.0895 \cdot 1/2$$
$$= 0.0448$$

5 $P(X=k) = \begin{cases} ck^2 & \text{if } k=1,2,3,4 \\ 0 & \text{else} \end{cases}$

a Calculate constant c

$$1 = \Sigma_k P(X=k)$$
$$= c\cdot 1^2 + c\cdot 2^2 + c\cdot 3^2 + c\cdot 4^2$$
$$= 30\cdot c$$
$$c = 1/30$$

b Calculate $E[X]$, $Var(X)$

$$E[X] = \Sigma_k k P(X=k)$$
$$= \Sigma_{k=1}^{4} k\cdot \tfrac{1}{30}k^2$$
$$= \tfrac{1}{30}(1^3+2^3+3^3+4^3)$$
$$= 100/30$$

$$Var(X) = E[X^2] - (E[X])^2$$
$$E[X^2] = \Sigma_k k^2 P(X=k)$$
$$= \Sigma_{k=1}^{4} k^2 \tfrac{1}{30}k^2$$
$$= \tfrac{1}{30}(1^4+2^4+3^4+4^4)$$
$$= 354/30$$
$$Var(X) = 354/30 - (100/30)^2$$
$$= 31/45$$

6 There are average 4.5 burglaries in some neighborhood. X = number of burglaries in some month.

If there are many houses in this neighborhood, each with a small independent chance of being burglarized, then it is reasonable to model X as a sum of low prob. indep. Bernoulli RV's. Whenever this is the case, Poisson approximation applies.

Thus $P(X=k) \approx e^{-\lambda} \lambda^k / k!$
$= e^{-4.5} \, 4.5^k / k!$

$$P(X > 10) = 1 - P(X \leq 10)$$
$$\approx 1 - e^{-4.5}\left(\frac{4.5^0}{0!} + \frac{4.5^1}{1!} + \dots + \frac{4.5^{10}}{10!}\right)$$

$$\approx 0.0067$$